A Method for Reduced-Order Modeling and Simulation of Large Interconnect Circuits and its Application to PEEC Models with Retardation

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***Abstract*—The continuous improvement in the performance and the increases in the sizes of VLSI systems make electrical interconnect and package (EIP) design and modeling increasingly more important. Special software tools must be used for the design of high-performance VLSI systems. Furthermore, larger and faster systems require larger and more accurate circuit models. The partial element equivalent circuit (PEEC) technique is used for modeling such systems with three-dimensional full wave models. In this paper, we present a practical, readily parallelizable procedure for generating reduced-order frequency-domain models from general full wave PEEC systems. We use multiple expansion points, and piecemeal construction of pole-residue approximations to transfer functions of the PEEC systems, as was used in the complex frequency hopping algorithms. We consider general, multiple-input/multiple-output PEEC systems. Our block procedure consists of an outer loop of local approximations to the PEEC system, coupled with an inner loop where an iterative model-reduction method is applied to the local approximations. We systematically divide the complex frequency region of interest into small regions and construct local approximations to the PEEC system in each subregion. The local approximations are constructed so that the matrix factorizations associated with each of them are the size of the original system and independent of the order of the approximation. Results of computations on these local systems are combined to obtain a reduced-order model for the original PEEC system. We demonstrate the usefulness of our approach with three interesting examples.**

***Index Terms*—Arnoldi, electronic interconnect package (EIP), interconnect circuits, iterative methods, Lanczos, MIMO systems, model reduction, reduced models, reduced-order systems, PEEC systems, time delays, transfer function.**

I. Introduction

HIGH-performance integrated circuit systems provide infrastructure to a multitude of applications. The prediction of the behavior and of the improvement of the performance of integrated circuit systems is becoming a problem of ever-increasing importance. Interconnects at all levels of the package are limiting the speed of the systems. The most difficult problems occur for the highest performance designs with clock rates of several hundred megahertz and/or signal rise times of less than 500 ps. The spectra of the time waveforms may extend from very low frequencies to above 10 GHz. Increased speed leads to an increase in signal couplings and in the geometric distances for which couplings must be taken into account. Both lead to commensurate complications in the electrical interconnect and package (EIP) and EMI modeling problem. Two other key application areas for EIP modeling techniques are mobile computing and communication circuits.

For many problems, the three-dimensional (3-D) nature of the interconnects and the electrical couplings limit the application of the usual two-dimensional (2-D) differential equation $\{L, R, C\}$, transverse electromagnetic waves (TEM) or quasi-TEM mode lumped circuit models [1]. Therefore, we consider 3-D partial element equivalent circuit (PEEC) models with losses (R) and retardation (τ) [2] which are capable of modeling these effects. These full wave models support all types of wave propagation modes, e.g., TEM, transverse electric (TE), and transverse magnetic (TM). We use the notation (L_p, P, R, τ)PEEC to indicate which types of circuit elements are included in the PEEC model. L_p represents the partial inductances, P the coefficients of potential, and R the series resistances. τ represents retardation (time delays). PEEC models result in very large dense nonsymmetric systems of delay-differential-algebraic equations corresponding to a wide range of 3-D conductor geometries.

In general, EIP analysis is required for a variety of geometries on a chip and also at several levels of a circuit package. Unfortunately, the complexity of the resultant models is such that an almost unlimited number of circuit elements can be generated if an accurate analysis needs to be performed. Hence, it is very desirable to reduce the interconnect model complexity before incorporating the effects of the interconnects into conventional circuit simulation techniques such as those employed in tools like SPICE [3]. Using model-reduction (MOR) techniques, much simpler macromodels can be obtained which can then be used in a multitude of applications. Reduced-order models must be constructed in such a way as to retain the important features of the original system. MOR for circuit applications was first introduced in [4]. Since then, many techniques have been proposed, primarily for classical circuits where the retardations (delays) are zero or where simple models suffice. Several researchers have applied MOR techniques to examples of quasistatic (L_p, P, R)PEEC models without retardation. See, for example, [5]–[7].

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Delay or retardation has only recently become an important issue for PEEC circuit modeling. Quasistatic models were sufficient for lower performance circuits. Procedures for the reduction of (L_p, P, R, τ) PEEC models with retardation were first proposed in [8], [9]. Both of these references construct reduced-order models for the frequency domain transfer function. For PEEC models with retardation the transformed system matrix $A(s)$ contains many elements with factors of the form$\exp(-s\tau)$ for some τ corresponding to a time delay (retardation) in the circuits. All of the those papers expand each delay term in an infinite Taylor series. Reference [8] uses a single expansion point $s = 0$ and applies asymptotic waveform (AWE) model reduction [4] to the resulting infinite order linear system. That paper includes an example for which good approximations were obtained for low frequencies up to 1 GHz. Reference [9] uses complex frequency hopping (CFH), considers several expansion points, applies AWE to the infinite linear system obtained at each such point, and then combines the pole and residue information to obtain an approximation to the transfer function of the PEEC system. It includes examples to demonstrate the use of this method to approximate the transfer function of a system up to 4.8 GHz. A similar approach has been applied to an EIP problem using a (L_p, P, R, τ)PEEC model in conjunction with a fast Fourier transform (FFT) grid representation [10]. Reference [10] considers a PEEC formulation and approximates that dense system by a related sparse system which can be used in the model reduction. Reference [10] also proposes using an iterative method to compute the matrices needed in the model-reduction procedure. More comments regarding differences between the MOR procedure proposed in Section VI, and the procedure in [10] are given in Section VI.

In this paper, we introduce a general MOR procedure for generating reduced-order frequency domain models of the transfer function of full-wave (L_p, P, R, τ) PEEC systems with retardation. Our block MOR procedure uses multiple expansion points, and piecemeal construction of a pole-residue approximation to the transfer function, as used for example in [9]. Our procedure consists of an outer loop of local approximations to the PEEC system, coupled with an inner loop where an iterative MOR method is applied to each local approximation. In contrast to previous work for (L_p, P, R, τ) PEEC, our local approximations are constructed so that the matrix factorizations required at each expansion point are the size of the original system and independent of the order of the approximation. Moreover, we systematically divide the user-specified complex frequency region of interest into smaller regions and construct local approximations to the PEEC model in each subregion. We demonstrate the usefulness of our approach with three interesting applications. (An abbreviated version of this paper appeared in [11]).

In the numerical examples presented, we obtain frequency domain characterizations of PEEC systems in terms of an impedance matrix description $V = ZI$. V represents port voltages and I represents terminal currents. In these examples, we were able to obtain better results using Z than using an admittance matrix Y representation. If the reduced-order model is employed as a macromodel in the frequency domain within a program based upon an MNA formulation [12], then the reduced-model can be stamped into the impedance part of the MNA matrix. This is similar to [13], [14] and other work where the Y matrix is stamped in the admittance part of the MNA matrix.

In Section II, we review PEEC models. The model-reduction procedure obtains a reduced-order model of the frequency domain transfer function of a general PEEC system. Therefore, in Section III, we review transfer functions of linear systems and their pole-residue representations. In Section IV, we review a model-reduction procedure for systems with no retardation which is based upon an Arnoldi recursion. The inner loop of our procedure requires an iterative method for obtaining reduced-order models of systems with no retardation.

Our model-reduction procedure divides the region of interest in the complex s-plane into smaller regions. Within each small region, we compute a local approximation to the original PEEC system and apply a MOR iterative method to that approximation. Our procedure extracts portions of each local reduced-order model and combines them to obtain a reduced-order model for the original PEEC system. The local approximations are discussed in Section V, where we relate these approximations to methods for solving nonlinear eigenvalue problems. The ideas in Sections III –V are combined in Section VI to obtain the new method for computing reduced-order models for full (P, L, R, τ) PEEC models with retardation. This method is not, however, specific to this particular application. In Section VII, we illustrate the successful use of this method to obtain reduced-order models for three interesting applications. We use the following notation and definitions.

A. Notation

A	(a_{ij}), $1 \le i, j \le n, n \times n$ matrix
$\lambda_j(A)$	$1 \le j \le n$, eigenvalues of A,$\omega(A)$, spectrum of A
$A(s)$	matrix function of complex variable s
$H(s)$	transfer function of a system
s	complex variable
$\mathcal{K}_j(A, b)$	span$\{b, Ab, \cdots, A^{j-1}b\}$, jth Krylov subspace for $\{A, b\}$
v_j	jth vector in any sequence of vectors, $V_j = \{v_1, \cdots, v_j\}$
p_j, r_j, z_j	poles, residues, zeros of system
I_j	$j \times j$ identity matrix
H_{Aj}	Arnoldi matrix generated using matrix A
si/so	system with a single input and a single output
mi/mo	system with more than one input or output

II. PEEC Models

In this section, we give an overview of a PEEC model formulation as a system of algebraic and delay-differential differential equations.

A. Mixed-Potential Integral Equation Formulation

For simplicity, we consider the case where the dielectric constant is either unity or the entire space is filled with a uniform dielectric. However, our MOR procedure and our codes include PEEC models with finite dielectrics [15]. Key aspects of the

PEEC formulation are the resulting conventional voltage and current variables and circuit elements which are used instead of electric and magnetic fields. The total electric field in a conductor is given by

$$\overline{E}_0(\overline{r},t) = \frac{\overline{J}(\overline{r},t)}{\sigma} + \frac{\partial \overline{A}(\overline{r},t)}{\partial t} + \nabla\Phi(\overline{r},t). \tag{2.1}$$

In (2.1), $\overline{E}_0$ denotes a potential applied electric field, $\overline{J}$ is the current density in the conductor, and $\overline{A}$ and Φ are respectively, the vector and scalar potentials corresponding to the magnetic and the electric fields associated with the conductor with a conductivity σ. The vector potential $\overline{A}$ of a single conductor at any point $\overline{r} = (x,y,z)$ is given by

$$\overline{A}(\overline{r},t) = \mu \int_{\acute{v}} G(\overline{r},\acute{\overline{r}})\, \overline{J}(\acute{\overline{r}},t_d)\, d\acute{v} \tag{2.2}$$

where $\acute{v}$ is the volume of material in which the current density is flowing and μ is the permeability of free space. The retarded time t_d is given by

$$t_d = t - \frac{|\overline{r} - \acute{\overline{r}}|}{c} \tag{2.3}$$

where c is the speed of light. The time delay equals the free space travel time between the points $\overline{r}$ and $\acute{\overline{r}}$. The corresponding Green's function is given by

$$G(\overline{r},\acute{\overline{r}}) \equiv \frac{1}{4\pi}\, \frac{1}{|\overline{r} - \acute{\overline{r}}|}. \tag{2.4}$$

Similarly, the scalar potential

$$\Phi(\overline{r},t) \equiv \frac{1}{\epsilon_0} \int_{\acute{v}} G(\overline{r},\acute{\overline{r}})\, q(\acute{r},t_d)\, d\acute{v}, \tag{2.5}$$

where ϵ_0 is the dielectric constant in free space, and q is the charge density on the surface. If the external electric field E_0 is zero and we substitute (2.2) and (2.5) into (2.1), we obtain (2.6) in terms of the current density and surface charge q

$$\frac{\overline{J}(\overline{r},t)}{\sigma} + \mu \int_{\acute{v}} G(\overline{r},\acute{\overline{r}})\, \frac{\partial \overline{J}(\acute{\overline{r}},t_d)}{\partial t}\, d\acute{v} + \frac{\nabla}{\epsilon_0} \int_{\acute{v}} G(\overline{r},\acute{\overline{r}})\, q(\acute{\overline{r}},t_d)\, d\acute{v} = 0. \tag{2.6}$$

Particular discretizations of the conductor geometry combined with the integral equation (2.6), lead to PEEC models with conventional circuit elements. This is accomplished by the breakup of the geometry into 2-D or 3-D cells, the integration of (2.6) over each resulting conductor volume cell [2], and observing that the first term in (2.6) corresponds to a resistor, the second term can be transformed into partial inductances [16], and the last term corresponds to normalized coefficients of potential [17]. In terms of circuit variables, the current is related to J through the cell cross-section. The charge density q is related to the total cell charge Q via the cell area, and the coefficients of potential P are related to the potential with$\Phi = PQ$. The unusual aspect of the PEEC models, from a circuit theory point of view, is the delay or retardation of signals among the circuit elements.

The electromagnetic fields couple the circuit elements. In PEEC models, the couplings between inductive circuit elements are represented by specifying mutual partial inductances for each pair of nonorthogonal self partial inductances. This leads to inductive coupling terms of the form

$$Lp_{\alpha\alpha}\dot{I}^L_\alpha(t) + Lp_{\alpha\beta}\dot{I}^L_\beta(t - \tau^L_{\alpha,\beta}) \tag{2.7}$$

whereI^L_α denotes the current through the self partial inductance $Lp_{\alpha,\alpha}$ while $Lp_{\alpha,\beta}$ represents the mutual partial inductance of cell α to cell β due to the current through $Lp_{\beta,\beta}$ in cell β. The physical distance between cells α and β results in an inductive delay or retardation represented by $\tau^L_{\alpha,\beta}$.

The couplings between the capacitive circuit elements are handled in a similar way. For most cases, capacitive coupling is implemented by introducing dependent currents sources which consist of scaled sums of the currents through all the self capacitances with the couplings delayed in time. This yields capacitive coupling terms of the form

$$\frac{1}{P_{\alpha\alpha}}\, \dot{V}^C_\alpha(t) - \frac{P_{\alpha,\beta}}{P_{\alpha,\alpha}}\, I^C_\beta(t - \tau^C_{\alpha,\beta}) \tag{2.8}$$

where V^C_α and I^C_β represent respectively the voltage at the self capacitance $1/P_{\alpha,\alpha}$ and the current through self partial capacitance $1/P_{\beta,\beta}$. The cell α has a nonretarded capacitance $1/P_{\alpha,\alpha}$ to infinity and $P_{\alpha,\beta}/P_{\alpha,\alpha}$ represents the mutual coupling of cell α to a cellβ from the capacitive current in cell β. The physical capacitive delay between cells α and β is represented by $\tau^C_{\alpha,\beta}$.

The resistances account for the physical resistance of the wires. The PEEC model automatically includes the continuity equation $\nabla \cdot \overline{J} + \dot{q} = 0$ in the MNA formulation as Kirchoff's current law at each PEEC node of the circuit. For (L_p, P, R, τ) PEEC models with retardation, the corresponding MNA equations reduce to systems of linear, time-invariant, delay-differential-algebraic equations

$$C\dot{x}(t) + \sum_{k,m=1}^{M} \tilde{C}_{k,m}\dot{x}(t - \tau^L_{k,m}) = -Gx(t) - \sum_{i,j=1}^{J} \tilde{G}_{i,j}x(t - \tau^P_{i,j}) + Bu(t). \tag{2.9}$$

The state variables x are the nodal voltages and the currents for each cell in the geometric conductor configuration. $\overline{G}_{i,j}$,$\overline{C}_{k,m}$ are coefficient matrices corresponding respectively to state and to derivative of state terms.

It is clear that practical systems possess a large number of delays, $\tau^L_{k,m}$,$\tau^P_{i,j}$, corresponding to each mutual partial inductance and to each coefficient of potential. These delays occur densely, in the sense that the PEEC model incorporates pairwise couplings between all coefficients of potential and between all nonorthogonal self partial inductances. A schematic of a simple two-cell PEEC model is given in Fig. 1. The diamond-shaped dependent current sources represent capacitive coupling terms of the type shown in (2.8). Not depicted in Fig. 1 are the mutual inductance couplings specified in (2.7). More details on the derivation can be found in, for example, [2], [18].

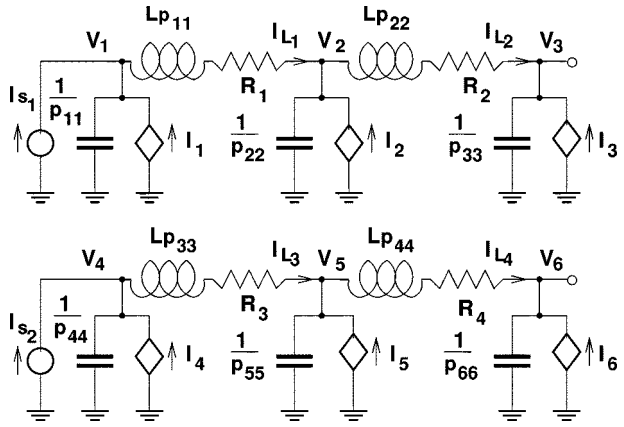


Fig. 1. PEEC circuit: two wires, two cells in each wire.

III. Transfer Functions of Linear Systems

In a practical situation, we are interested in the response of a conductor configuration to various inputs applied at specified locations in the configuration. Responses of the system are measured at the same or different specified locations. For example, an input could be a current source which is applied between two nodes in the PEEC model, and an output could be the difference in voltages across pairs of terminals. Typically, we are interested in several different combinations of inputs and outputs.

In studying linear systems, it is convenient to work in the frequency domain. If we apply Laplace transforms to (2.9) and add the relationship between the system solution and the output vectors of interest, we obtain the following complex transcendental system

$$\begin{aligned} Gx(s) + \sum_{i,j=1}^{J} \tilde{G}_{i,j}x(s)\exp(-\tau_{i,j}^{P}s) + sCx(s) \\ + \sum_{k,m=1}^{M} s\tilde{C}_{k,m}x(s)\exp(-\tau_{k,m}^{L}s) = Bu(s), \\ y(s) = Ex(s). \end{aligned} \tag{3.10}$$

Equation (3.10) can be rewritten in the following form, where $A(s)$ is the matrix obtained by combining all of the matrix terms

$$\begin{aligned} A(s)x(s) &= Bu(s) \\ y(s) &= Ex(s). \end{aligned} \tag{3.11}$$

$A(s)$ is a transcendental matrix function of s, the columns of B place the specified sources (inputs) in the correct equations, and the rows of E select the combinations of variables (outputs) which are of interest to the user.

The transfer function of a system is the mapping$H(s)$ from the inputs $u(s)$ to the outputs $y(s)$ [19]. Specifically

$$y(s) = H(s)u(s), \quad \text{where} \quad H(s) \equiv EA(s)^{-1}B. \tag{3.12}$$

If the system has a single input and single output, and $A(s)$ is a linear function of s, then $H(s)$ is a rational function of s. Formi/mo systems, $H(s)$ is a $p \times q$ matrix. Each entry $(H(s))_{i,j}, l \le i \le p, 1 \le j \le q$, is the si/so transfer function for the corresponding ith output and jth input pair [19]. In PEEC models, $A(s)$ is a matrix where many entries are not only nonlinear but transcendental functions of s.

The behavior of any linear, time invariant system described by (3.10) can be studied through its transfer function. Each si/so transfer function, $(H(s))_{i,j}$. is completely characterized by its poles and zeros or by its poles and residues. Nominally, all $(H(s))_{i,j}$ have the same poles. Depending upon the inputs and outputs, it is possible for one or more poles to cancel with common zeros.

System poles p_i correspond to natural frequencies of the system, nontrivial solutions $x \neq 0$ of the matrix equations $A(p_i)x = 0$. System zeros z_j correspond to complex frequencies s which are hidden by the system. Outputs corresponding to an input at such a frequency are identically zero. Residues and zeros are dependent upon the specified inputs and outputs. A system will be said to be stable if all of the solutions of the system are bounded over time. Stability requires that all of the poles of the system be in the closure of the left-half complex plane and that any poles which are on the imaginary axis be simple.

For any si/so system, we have the following two equivalent representations of the transfer function where g is a scale factor. For PEEC systems without retardation, the number of poles and zeros is finite

$$H(s) = g\frac{\prod_{i=1}^{\infty}(s - z_i)}{\prod_{j=1}^{\infty}(s - p_j)} = \sum_{j=1}^{\infty}\frac{r_j}{(s - p_j)}. \tag{3.13}$$

We use the pole-residue formulation in Section VI.

IV. Model Reduction—Iterative Arnoldi: No Time Delays

The model-reduction procedure described in Section VI consists of an outer loop, where local approximations to the original system are constructed, and an inner loop, where reduced-order models for each local approximation are generated. Each local approximation is transformed into a corresponding linear (in frequency) problem, and an iterative model-reduction procedure for systems with no time delays is applied to each transformed problem.

Any type of model-reduction procedure which generates pole and residue or pole and zero information can be used. In this section, we review an iterative Arnoldi model-reduction method. We have also used Lanczos methods and a two-sided Arnoldi procedure which possesses the characteristics of a Lanczos procedure, but translates readily to the block form needed for the mi/mo problems encountered in practice [20].

Pioneering work on using iterative methods for model reduction of $\{R, C, L\}$ circuits can be found in [21]–[24]. Reduced-order systems obtained using these techniques can be used to predict the time domain or the frequency-domain response of such systems over a prespecified range of excitation frequencies.

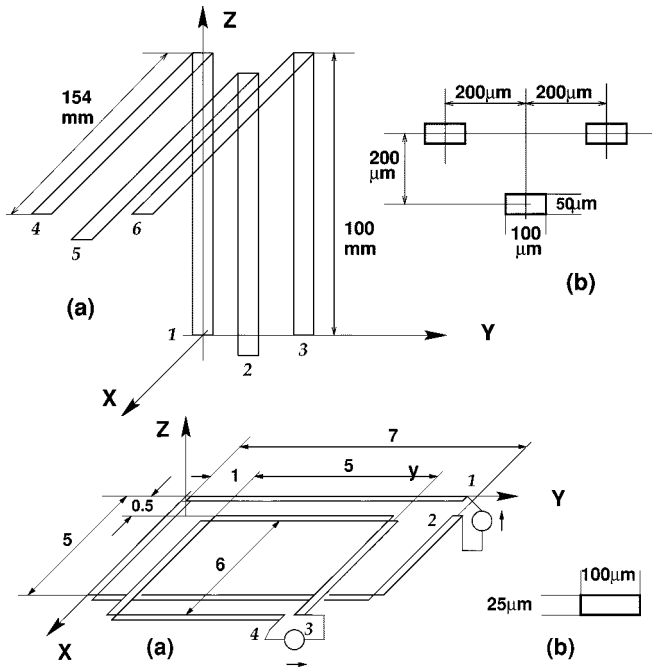


Fig. 2. Top: (a) three wires with bend and (b) cross-section. Bottom: (a) two loops and (b) cross-section 1w.

If the time delays in (3.10) are all zero, then $A(s) = sC + G$ and system (3.10) reduces to

$$\begin{aligned} sCx(s) &= -Gx(s) + Bu(s). \\ y(s) &= Ex(s). \end{aligned} \tag{4.14}$$

Typically, G is not invertible and we select an expansion point s_0 and rewrite (4.14) as

$$\begin{aligned} \sigma A\tilde{x}(\sigma) &= -\tilde{x}(\sigma) + R\tilde{u}(\sigma). \\ \tilde{y}(\sigma) &= E\tilde{x}(\sigma) = y(s) = Ex(s) \\ A &\equiv (s_0C + G)^{-1}C \\ R &\equiv (s_0C + G)^{-1}B \end{aligned} \tag{4.15}$$

where $s = s_0 + \sigma.s_0$ is chosen, such that $s_0C + G$ is nonsingular. The transfer function $H(s) = E(sC + G)^{-1}B = E(I + \sigma A)^{-1}R$.

In many systems, it is necessary to consider the system at a set of expansion points $s_k, k = 1, \cdots, K$ where K may be as few as 2 and to combine information collected for each s_k, as in [9]. In [25] the iterative method explicitly combines information garnered near each expansion point and produces a reduced-order model directly in terms of a reduced transfer function matrix, inputs, and outputs.

Formally, we can expand

$$H(s) = \sum_{k=0}^{\infty} EA^kR\sigma^k. \tag{4.16}$$

The terms EA^kR for $k = 0, 1, \cdots$ are called the moments of $H(s)$ with respect to R and E. Given a si/so system and m moments of the corresponding transfer function $H(s)$, where $m = 2 * M - 1$, we could ask for the Padé approximation to $H(s)$, which is the ratio of a s-polynomial of order $M - 1$ to a s-polynomial of order M. The asymptotic waveform evaluation (AWE) procedure [4] computes Padé approximations to $H(s)$, [26]. In practice, AWE can work well when the first few moments suffice. However, AWE is equivalent to the explicit computation of the effect of repeated powers of a system matrix on a particular vector, and its ability to approximate the relevant portion of the system may be adversely affected by the presence of dominant system poles.

The Padé via Lanczos (PVL) model-reduction algorithms described in [21], [22] are implicit implementations of AWE which obviate the problems with AWE. The Lanczos recursions compute moments of $H(s)$ as moments of a reduced system which is generated by the Lanczos recursions. In, fact the Lanczos algorithms generate Padé approximations to $H(s)$, as they simultaneously generate corresponding reduced-order systems. Extensions to mi/mo systems are discussed in [23].

Corresponding Arnoldi algorithms, described for example in [27], match moments but do not generate Padé approximations. In the numerical examples, we used a block version of recursion (4.17) and a new two-sided block Arnoldi procedure [20]. These two methods generated very similar approximations.

A. Arnoldi Recursion

The basic Arnoldi recursion [28] is a direct extension of the real symmetric Lanczos recursion to nonsymmetric matrices. We review briefly a si/so Arnoldi model-reduction algorithm to illustrate the ideas. In practice, the systems are mi/mo and block or band versions of these algorithms are used [29], [20].

Basic Arnoldi Recursion:

1) Specify $v_1 = b/\|b\|$ where b is the input vector.
2) For each $j = 2, 3, \cdots, M$, compute:
   (a) $v_{j+1} = Av_j$;
   (b) for $1 \le i \le j$, $h_{ij} = v_i^H v_{j+1}$, $v_{j+1} = v_{j+1} - h_{ij}v_i$;
   (c) $h_{j+1,\,j} = \|v_{j+1}\|$;
   (d) $v_{j+1} = v_{j+1}/h_{j+1,\,j}$.

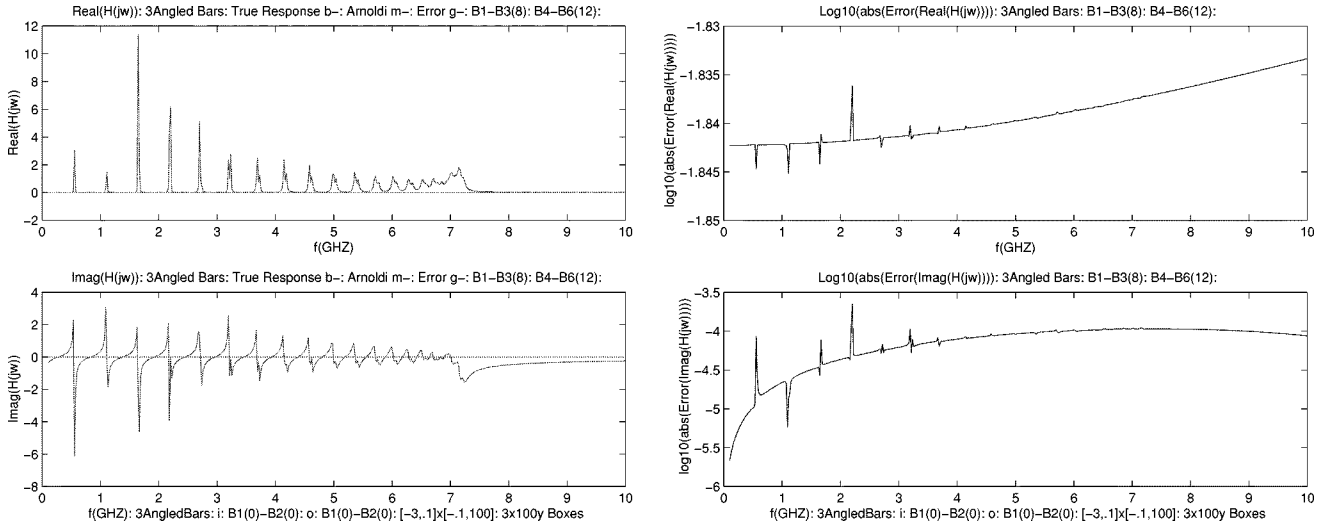
Given any $\{A, v_1\}$, the coefficients in the Arnoldi recursion are chosen so that the Arnoldi vectors $V_j \equiv \{v_1, \cdots, v_j\}, j = 1, 2, \cdots$ are an orthonormal basis for the Krylov subspace $\mathcal{K}_j(A, v_1)$. The corresponding Hessenberg matrix $H_{jA} \equiv V_j^H AV_j = (h_{ik}), 1 \le i,k \le j$ is a matrix representation of the orthogonal projection of A onto the Krylov subspace $\mathcal{K}_j(A, v_1)$. In matrix form, these recursions become

$$AV_j = V_jH_{jA} + h_{j+1,\,j}v_{j+1}e_j^T. \tag{4.17}$$

B. Model Reduction: $A(s)$ Linear in s

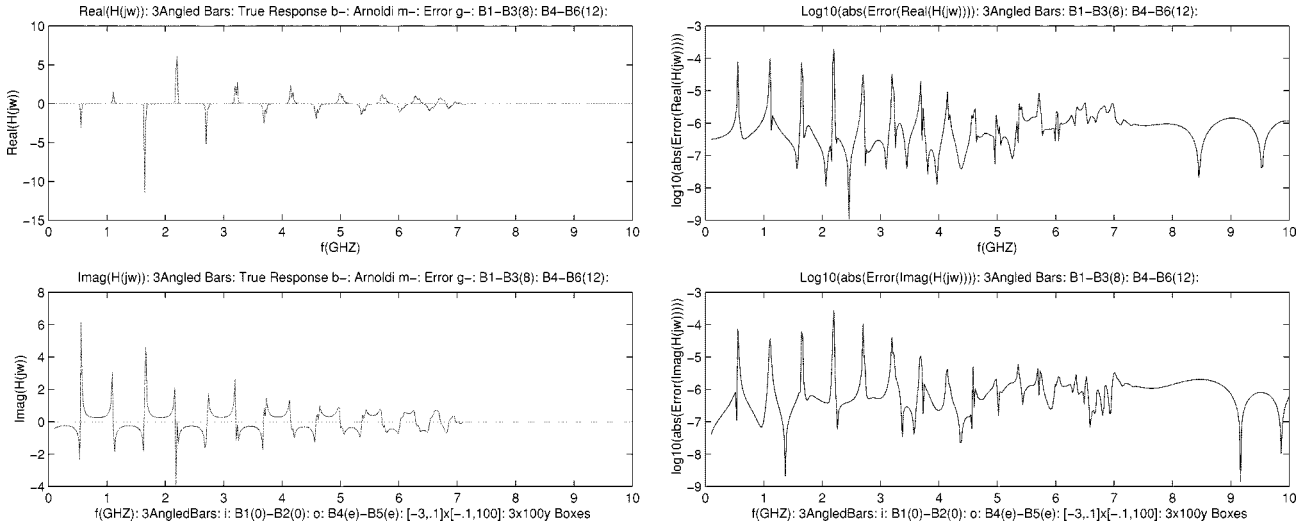
Since H_{jA} is an upper Hessenberg matrix, $H_{jA}^k e_1$ is in the space spanned by the coordinate vectors $e_m, 1 \le m \le k+1 \le j$. Using this fact and (4.17), it is straightforward to demonstrate that

$$A^kV_je_1 = V_jH_{jA}^k e_1, \quad \text{for } 0 \le k \le j-1. \tag{4.18}$$



(a) (b)

Fig. 3. Three wires. (a) True and computed self-impedance *1–2*. (b) $\log_{10}$[abs(Error)] for self-impedance.



(a) (b)

Fig. 4. Three wires. (a) True and computed trans-impedance *1–2; 4–5*. (b) $\log_{10}$[abs(Error)] for trans-impedance.

Formally, using (4.16) and (4.18), we observe that $V_j(I+\sigma H_{jA})^{-1}e_1$ provides an approximation to $(I+\sigma A)^{-1}v_1$. Therefore

$$\tilde{H}_j(\sigma) \equiv \|b\|EV_j(I+\sigma H_{jA})^{-1}e_1 \text{ approximates } H(s), \quad \text{where } s = s_0 + \sigma. \tag{4.19}$$

The Arnoldi vectors appear explicitly in (4.19). In contrast, if a nonsymmetric Lanczos recursion is used, the Lanczos vectors do not have to be saved [22]. The Arnoldi Hessenberg matrices H_{jA} can be used to construct corresponding-reduced-order systems

$$H_{jA}\dot{z} = -z + \|b\|e_1 u, \quad y = EV_j z. \tag{4.20}$$

The preceding computations are summarized in the following si/so Arnoldi model-reduction procedure.

Model-Reduction Procedure: $A(s) = Cs + G$*:*

1) Select an expansion point s_0, set $s = s_0+\sigma$ and transform system (4.14) to system (4.15).
2) Apply the Arnoldi recursion to obtain a reduced-order system $\{z, H_{jA}, \|b\|e_1, EV_j\}$.
3) Use this system to compute an approximation $\tilde{H}_j(\sigma)$ to the transfer function of the original system.

C. Poles, Eigenvalues, and Residues

In practice, one expansion point is not sufficient to characterize the system. Computations are performed at different expansion points which ideally are chosen to emphasize the desired dominant part of the spectrum of A**.** The iterative Lanczos model-reduction procedure described in [24] is an analog of an Arnoldi procedure proposed in [30]. It combines the results of one or more successive applications of the Lanczos method at different expansion points into a pair of reduced-order matrices which are used to define a reduced-order model of the original

system. Ideally these shifts are chosen to emphasize the desired dominant part of the spectrum of A. In practice, shift selection is a nontrivial exercise.

The approaches in [30], [24] require linear $A(s)$ and are not directly applicable to PEEC models. Alternatively, we could use a pole/residue formulation for the transfer function. This would allow us to specify a set of expansion points, expand independently at each such point, compute poles and residues near each expanded point, and then sum the local information. This approach was used successfully in [9]. Formulation is suitable for PEEC models.

The poles of any system with $A(s) = G + sC$ correspond to eigenvalues of the generalized linear eigenvalue problem

$$-Gx = sCx \tag{4.21}$$

and we can consider using an Arnoldi method to obtain good approximations to the poles of such systems. If the number of inputs equals the number of outputs, the zero computations can also be formulated as eigenvalue problems [19].

Residues corresponding to computed poles can be obtained using relationship (4.19). We assume that H_{jA} is diagonalizable. That is, $H_{jA} = Z_j \Theta_j W_j^T$, where each column z_{jk} of Z_j is a right eigenvector of H_{jA} corresponding to the eigenvalue θ_{jk}, and each column w_{jk} of W_j is a corresponding left eigenvector. The approximation to the transfer function of the original system

$$\hat{H}_j(s) = \tilde{H}_j(\sigma) \equiv \|b\| E V_j \sum_{k=1}^{j} \frac{z_{jk} w_{jk}^T}{1 + \sigma\theta_{jk}}\, e_1. \tag{4.22}$$

The corresponding approximation r_{jk} to the residue for the pole $p_{jk} \equiv s_0 - 1/\theta_{jk}$ is defined as

$$r_{jk} = \|b\| E V_j z_{jk} w_{jk}^T e_1 / \theta_{jk}. \tag{4.23}$$

If there is more than one input, e_1 in (4.23) is replaced by $E_1 T$, where E_1 is the $[1:j, 1:q]$ submatrix of I_j, and $B = V_1 E_1 T$, with V_1 the first block in a block Arnoldi variant of the procedure described in Section IV.

Using this approach we obtain the following modified Arnoldi model-reduction procedure.

Model-Reduction Procedure: $A(s) = Cs + G$: *Poles/Residues:* For each expansion point $s_k, k = 1, 2, \cdots, K$, do the following:

1) Expand system (4.14) about point s_k to obtain system (4.15).
2) Apply the Arnoldi recursion to system (4.15) to compute approximations to some poles and residues of this system.
3) Select subsets of the computed poles and residues to accumulate for use in a pole-residue model for the original system.

V. Local Approximations

In this section, we provide justification for the particular local approximations which we use in our MOR procedure described in Section VI. The arguments used to justify the use of the iterative MOR procedures summarized in Section IV require that $A(s)$ is a linear matrix function of s. In PEEC models $A(s)$ is a transcendental matrix function of s. Therefore, we cannot directly use those methods. If $A(s)$ were a polynomial matrix function of s,

$$A_k(s) = M_k s^k + M_{k-1} s^{k-1} + \cdots + M_0 \tag{5.24}$$

we could reduce the nonlinear pole computations, $A(s)x = 0$, for the system defined by (3.11) to the following larger but linear generalized eigenvalue problem.

$$\begin{bmatrix} M_0 & M_1 & M_2 & M_3 & & \cdots & M_{k-1} \\ 0 & I & 0 & 0 & & \cdots & 0 \\ 0 & 0 & I & 0 & & \cdots & 0 \\ & & & & \cdots & & \\ & & & & \cdots & & \\ 0 & 0 & & & & & I \end{bmatrix} \begin{bmatrix} x \\ z_2 \\ \vdots \\ \vdots \\ z_k \end{bmatrix}$$
$$= s \begin{bmatrix} 0 & 0 & 0 & & & \cdots & -M_k \\ I & 0 & 0 & 0 & & \cdots & 0 \\ 0 & I & 0 & 0 & & \cdots & 0 \\ & & & & \cdots & & \\ & & & & \cdots & & \\ 0 & 0 & & & & I & 0 \end{bmatrix} \begin{bmatrix} x \\ z_2 \\ \vdots \\ \vdots \\ z_k \end{bmatrix} \tag{5.25}$$

Denote the left-hand matrix in (5.25) by C_k and the right-hand matrix by D_k. If M_0 is invertible, C_k is invertible. Regardless of the size of the order k, the computation of C_k^{-1} requires only the computation of M_0^{-1}, a matrix whose size is the size of the original system. Specifically, we have (5.26), shown at the bottom of the page.

In this situation, the pole computations in (5.25) reduce to the following standard eigenvalue problem:

$$L_k z = -z/s \tag{5.27}$$

where we have (5.28), shown at the bottom of the next page. Therefore, the pole computations for polynomial systems reduce to pole computations of larger linear systems. We have the following Theorem which when $A(s)$ is a polynomial function of s, justifies the replacement of (3.11) by a related linear system

$$C_k^{-1} = \begin{bmatrix} M_0^{-1} & M_0^{-1}M_1 & M_0^{-1}M_2 & M_0^{-1}M_3 & & \cdots & M_0^{-1}M_{k-1} \\ 0 & -I & 0 & 0 & & \cdots & 0 \\ 0 & 0 & -I & 0 & & \cdots & 0 \\ & & & & \cdots & & \\ & & & & \cdots & & \\ 0 & 0 & & & & & -I \end{bmatrix} \tag{5.26}$$

(5.30), defined by L_k in (5.28). Theorem 5.1 enables us to apply standard iterative model-reduction methods as discussed in Section IV to systems defined by (3.11) when $A(s)$ is a polynomial function of s.

Theorem 5.1: Let $A_k(s) = M_k s^k + M_{k-1}s^{k-1} + \cdots + M_0$ and consider the system

$$\begin{aligned} A_k(s)x &= Bu \\ y &= Ex. \end{aligned} \tag{5.29}$$

If M_0 is invertible, the transfer functions of the nonlinear polynomial system (5.29) and of the larger linear system (5.30) are identical. Given any solution x of the nonlinear equation in (5.29), the $kn \times 1$ function z whose $[(j-1)n+1) : jn]$ components$z_j \equiv s^{j-1}x$ for $1 \le j \le k$, is a solution of the linear equations in (5.30). Conversely, given any solution z of the linear equations in (5.30), $x \equiv z_1$ is a solution of the nonlinear equation in (5.29). Moreover, for any input$u(s)$, $y_{L_k}(s) \equiv E_k(I + sL_k)^{-1}R_k u(s) = y_{A_k(s)}(s) \equiv EA_k(s)^{-1}Bu(s)$

$$\begin{aligned} z + sL_k z &= R_k u \\ y &= E_k z \end{aligned} \tag{5.30}$$

where

$$R_k = \begin{bmatrix} M_0^{-1}B \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_k^T = \begin{bmatrix} E^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}. \tag{5.31}$$

Proof 5.1: The proof follows by direct substitution and the observation that $y_{L_k}(s) = E_k z(s) = Ez_1(s)$.

Corollary 5.1: Under the hypotheses of Theorem 5.1, the assumption that L_k is diagonalizable, and that all of the poles are simple, the transfer functions of the systems defined by (5.30) and by (5.29) have the same poles and matrix residues.

Proof 5.2: From Theorem 5.1 we know that the transfer functions of these two systems are identical. Any polep_m of system (5.30) corresponds to some eigenvalue, eigenvector pair, $\{\lambda, z\}$, where $\lambda L_k z = -z$. Since $z_j = \lambda^{j-1}z_1$ for $1 \le j \le k$, the pair $\{\lambda, z_1\}$ clearly satisfies $A_k(\lambda)z_1 = 0$. Conversely, given $\{\lambda, x\}$ satisfying $A_k(\lambda)x = 0$, if we define z such that $z_j = \lambda^{j-1}x$ then $\{\lambda, z\}$ satisfies $z + \lambda L_k z = 0$. Therefore, the two systems have identical poles.

Let $L_k = W\Theta U^T$ be a diagonalization ofL_k. The transfer function of system (5.30)

$$\begin{aligned} H_{L_k}(s) &= E_k(I + sL_k)^{-1}R_k \\ &= E_k \sum_{m=1}^{kn} \frac{w_m u_m^T}{(1 + s\theta_m)} R_k. \end{aligned} \tag{5.32}$$

But

$$H_{A_k}(s) = H_{L_k}(s). \tag{5.33}$$

Using the pole-residue formulations and knowing that the two systems have the same poles, we can conclude that they must also have identical residues.

PEEC systems with retardation are transcendental, not polynomial systems. Therefore, the equivalence demonstrated in Theorem 5.1 is not directly applicable to them.

To obtain a model-reduction procedure for PEEC and other systems with time delays, we could consider replacing each $\exp(-\tau s)$ in $A(s)$ by a Padé approximation. This approach has been used successfully in the control systems literature and may be suitable for systems with a very few time delays. However, the time delays in PEEC models are a consequence of the travel times of waves between pairs of cells used in the model. The number of different delays depends upon the numbers of cells used in the model, and the sizes of the delays depends upon the geometry of the conductor configuration. Since travel times between every pair of cells are incorporated, there are numerous delays and the delay terms constitute dense portions of the matrix $A(s)$.

Therefore, we take a different approach. Physical limitations on the PEEC system restrict the range of complex frequencies of interest in the complex plane. We can specify a region in the complex plane which encompasses all of the complex frequencies of interest to the given system. We then subdivide that region and construct a polynomial approximation to our original PEEC system in each subdivision. In the discussion we assume the region is a box in the complex plane and that the subdivisions are smaller boxes with center points s_k, $k = 1, \cdots, M$.

Our constructions require approximations to the exponential terms in $A(s)$ which correspond to the time delay terms. For any expansion point s_0 and $s = s_0 + \sigma$ we have $e^{-\tau s} = e^{-\tau s_0}e^{-\tau\sigma}$. Therefore, individually on each exponential time delay entry, and for some specified J,, we make the following approximations:

$$e^{-\tau s} \approx e^{-\tau s_0}[1 - \tau\sigma + \tau^2\sigma^2/2 - \cdots - (-1)^J\tau^J\sigma^J/J!]. \tag{5.34}$$

$$L_k \equiv C_k^{-1}D_k = \begin{bmatrix} M_0^{-1}M_1 & M_0^{-1}M_2 & M_0^{-1}M_3 & \cdots & M_0^{-1}M_k \\ -I & 0 & 0 & \cdots & 0 \\ 0 & -I & 0 & \cdots & 0 \\ & & & \cdots & \\ & & & \cdots & \\ 0 & 0 & & -I & 0 \end{bmatrix}. \tag{5.28}$$

PEEC models contain two types of exponential delay terms: $f_1 = \exp(-\tau s)$ and $f_2 = s\exp(-\tau s)$. If, for example, we use a quadratic approximation to f_1 inside of f_2, we obtain the following approximation for f_2:

$$s\exp(-\tau s) \approx \exp(-\tau s_0)[s_0 + \sigma(1 - s_0\tau) - \sigma^2\tau(1 - s_0\tau/2 - \sigma\tau/2)]. \quad (5.35)$$

If we want to use a quadratic approximation to $A(s)$, we must drop the σ^3 term in (5.35). This corresponds to writing $f_2 = \exp(-s_0\tau)\,(s_0\exp(-\sigma\tau) + \sigma\exp(-\sigma\tau))$ and expanding the first term to second order and the second term to only first order.

Polynomial approximations of any order could be used. As defined in Theorem 5.1, the associated larger linear system defined by $\{L_k, B_k, E_k\}$ is nominally of size kn where k is the order of the polynomial approximation to $A(s)$. However, the expensive part of the L_k computations is the factorization of the matrix M_0, which is of the same size as the size of the original problem and is independent of the order of the approximation. Moreover, iterative Arnoldi model-reduction procedures on L_k, require only matrix-vectors multiplications $L_k z$. Since the $[(n+1) : nk]$ rows of L_k involve only identity matrices, the increases in the amount of computation required for increases in the order of the approximations are typically very small. In Section VII, we used only quadratic approximations $k = 2$.

The goodness of these approximations depends upon the sizes of the time delays, the sizes of the small boxes, and the order of the approximations.

The system described by (3.11) has an infinite number of poles. If all of the time delays were zero and the system was lossless, there would be a finite number of poles and they would lie on the imaginary axis. With nonzero time delays, these poles can move into either the left or the right half-plane. Examples of each case are given in [31].

VI. Model-Reduction Procedure for (L_p, P, R, τ)PEEC Models

We present a procedure for the model-reduction of general full wave (L_p, P, R, τ) PEEC systems. The procedure generates a sequence of local approximations to the original PEEC system, performs model-reduction on each local approximation, and then combines portions of the local reduced-order transfer functions to obtain an approximation to the transfer function of the original full wave (L_p, P, R) PEEC system. The procedure uses expansion points which are pre-specified by the user and yields an approximation for the (L_p, P, R, τ) PEEC model by combining pole and residue information obtained from the local approximations to the original system. The procedure can be summarized as follows.

(L_p, P, R, τ)PEEC Model-Reduction Procedure: Given a general PEEC system: $A(s)x = Bu, y = Ex$:

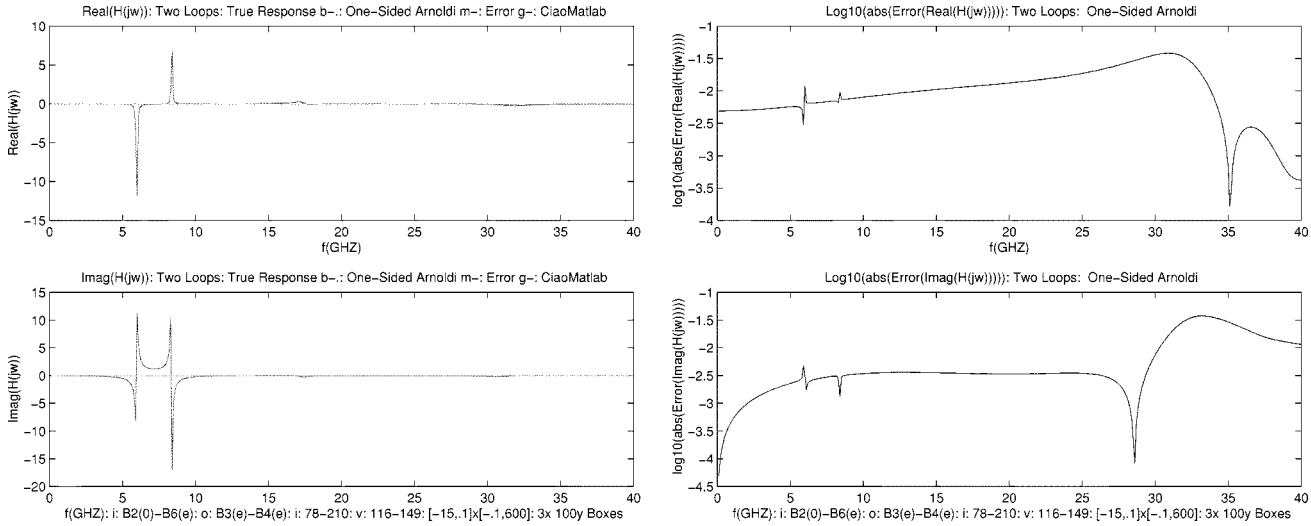
1) Specify the physically relevant region Ω in the complex s-plane and systematically subdivide Ω into smaller regions Ω_j, for $1 \le j \le K$.
2) For each $\Omega_j, j = 1, \cdots, K$
   (a) Construct a polynomial approximation $A_j(s)$ of order $k(j)$ to $A(s)$.
   (b) Construct the larger but linear system (5.30) obtained from $A_j(s)$, B, E and using $L_{k(j)}$.
   (c) Apply an iterative method to the system defined by $L_{k(j)}$ to obtain an approximation, $\tilde{H}^{L_{k(j)}}$, for the transfer function of that linear system.
3) Combine portions of each $\tilde{H}^{L_{k(j)}}(s)$ to obtain an approximation $\tilde{H}_{\text{approx}}(s)$ to the transfer function of the original (L_p, P, R, τ)PEEC system.

We are making two levels of approximations. We approximate the original transcendental system by a family of polynomial systems. We then apply a standard iterative model-reduction method to each member of this family to obtain a reduced-order model approximation to the transfer function of each member. If these polynomial problems are *good* approximations to the original system in their respective boxes, then for the frequencies within those boxes, the original system should have similar behavior. In particular, if any poles of the polynomial system corresponding to $L_{k(j)}$ lie in Ω_j, then we expect these quantities to approximate poles of the original system.

We systematically cover the user-specified region Ω to determine approximations to the poles (and residues) of the original system which are contained in each specified subregion. The only *a priori* restriction on these subdivisions is that the origin cannot be used as an expansion point. An approximation to the transfer function of the original system which simulates the behavior of the original system within the entire region Ω is then constructed by combining the local pole and residue information obtained in each subbox. Computations within different subboxes can be done in parallel.

Reference [10] also proposes a Krylov-based MOR procedure for PEEC models. In [10] the system matrix $A(s)$ is expanded in an infinite Taylor series. Using M_0^{-1}, a PEEC model $A(s)x = bu$ is transformed into an infinite linear system $(I_\infty + sL_\infty)z = M_0^{-1}b$. An Arnoldi recursion is applied to the L_∞ system. The authors indicate that several expansion points are considered and that ideas from CFH [9] can be used to determine which of the computed pole approximations should be used in the overall MOR approximation. The authors also consider a sparse but transcendental approximation to PEEC systems using a formulation for which all of the variables are currents. Two numerical examples are provided. The discussion in [10] does not, however, indicate how the infinite operator computations required for the Arnoldi method were implemented, nor how the expansion points were selected.

The PEEC MOR procedure described in this section provides a systematic procedure for MOR of general PEEC systems. Quadratic (second-order) polynomial approximations are used in the three examples presented in Section VII. The same order of approximation is used in each small box. Expansion points are simply the center points of each small box, avoiding the origin. Converged pole approximations are accepted only if



(a) (b)

Fig. 5. Two loops. (a) True and computed trans-impedance *1–2; 3–4*. (b) $\log_{10}$[abs(Error)] for trans-impedance.

they fall within the subbox currently being considered. The subboxes are not overlapped (although there is no *a priori* reason why they could not be overlapped). Different order Arnoldi approximations could be used in different subboxes. The procedure uses heuristics to decide when it has *found* all of the poles in a given subbox.

The matrix functions $A_j(s)$ of order $k(j)$ required by our MOR procedure were obtained by modifying IBM PEEC codes to generate directly new codes for computing $A_j(s)$ corresponding to any expansion point s_0. In the computations, M_0 was factored, and its inverse was computed from the factors. One could consider doing the inverse computation iteratively as suggested in [10].

We have not studied the relationships between the choice of subbox size, the choice of the order of the polynomial approximation, and the size of the time delays. The examples in Section VII indicate that it is not necessary to make $||A(s)-A_k(s)||$ very small, probably because the procedure is computing poles of transfer functions, points at which the transfer function is singular. These issues require further study.

We also note that the manner in which the approximate problems are formulated makes the pole computations correspond to computing eigenvalues of largest magnitude of associated matrices. Typically, these are the eigenvalues most readily computable by an Arnoldi procedure.

Our procedure can be applied to any PEEC system. It is readily parallelizable by subbox. The cost of the MOR should be somewhat more than the cost of one frequency-domain analysis for the original PEEC system. Runtimes are not provided because our codes are experimental with the focus on verifying the ideas in the MOR procedure and not on optimizing the computations. Memory usage is dominated by the factorization of the M_0 and the storage of the M_j. There is much redundancy in the M_j and that could be exploited to reduce both the storage and the computational costs but we have not yet done that. There are also open questions about using fast methods which have not yet been addressed.

VII. Examples of Reduction of (L_p, P, R, τ) PEEC Models

We consider three examples. In each case, the region of interest Ω was obtained using the maximum frequency of interest to the user. We used quadratic approximations to the original system, and applied Arnoldi's method to each localized linear system obtained by expanding the original system about the center of each subbox. Subbox sizes were determined using the largest time delays. For the first two problems, we generated the corresponding true transfer function and compared it with the approximations obtained using the MOR procedure defined in Section VI. The first example was chosen to illustrate the use of this method on problems with very large time delays.

As indicated in Section I, we consider the characterization of multi-port macromodels for PEEC circuits in terms of an impedance matrix formulation, $V(s) = Z(s)I(s)$. We apply current sources across pairs of nodes and measure voltage differences across the same or other nodes.

Example 1: Three Wires with Bend: We consider three long, parallel, bent wires terminating on the same vertical plane. See Fig. 2 (top). The time delays for this example are large, ranging up to 0.612 ns. We applied a 1-mA source current across terminals *1* and *2* and computed approximations to all transfer functions corresponding to the voltage differences across (1a) *1–2*, (1b) *4–5*, and (1c) *5–6*.

In this example, Ω was set equal to $[-3, 0.1] \times [-0.1, 100.]$. Each subbox was 1×1. Fig. 3(a) are plots of the real and the imaginary parts of the true transfer function for output (1a), together with the computed approximation to that transfer function, versus the frequency in gigahertz. Fig. 3(b) are corresponding plots of the $\log_{10} abs(Error)$. Fig. 4(a) and (b) are similar plots for output (1b). We observe that these errors are uniformly small.

Example 2: Two Loops: We consider two small rectangular loops separated vertically by 0.2 mm. See Fig. 2 (bottom). The time delays range up to 0.028 ns. We applied unit source currents

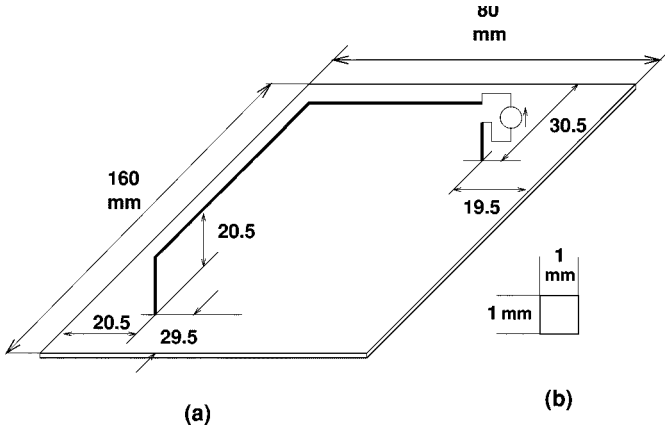


Fig. 6. Loop over ground plane.

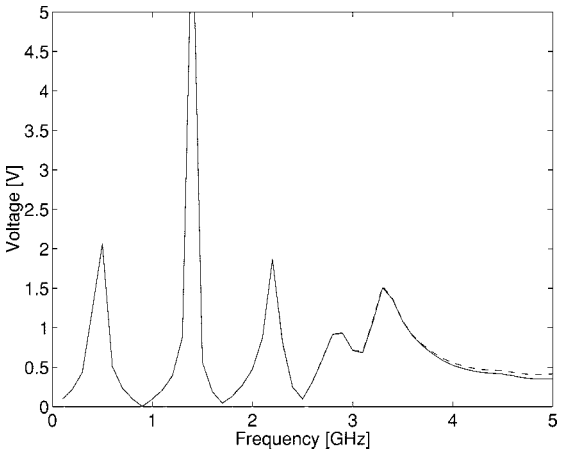


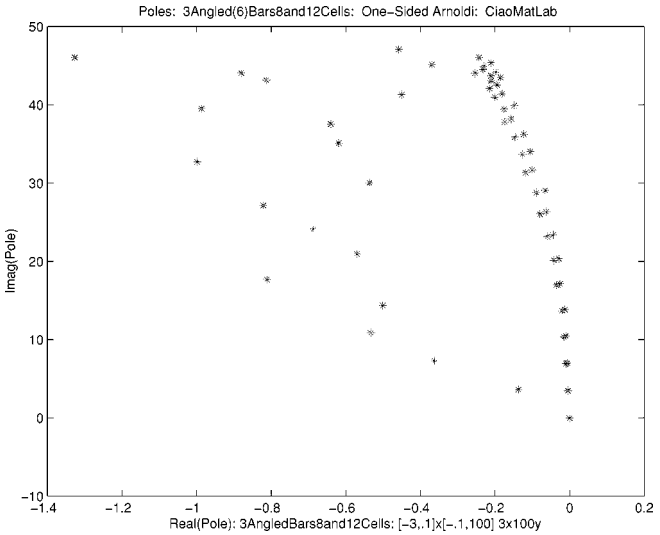
Fig. 7. Magnitude of output voltage for loop over plane. Frequency-domain analysis (−). MOR computation (−) .

across terminals (2ia) *1–2* and (2ib) *3–4* and then for each individual source computed the resulting voltage differences across terminals (2oa) *1–2* and (2ob) *3–4*.

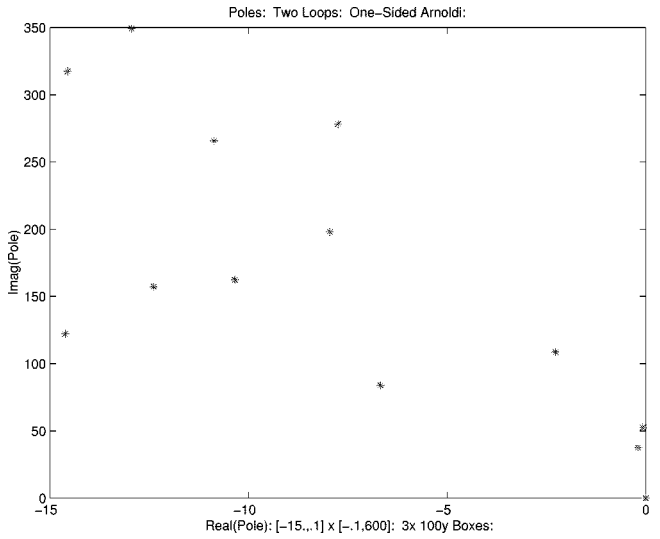
For this example, Ω was set equal to the box $[-15, 0.1] \times [-0.1, 600.]$. Each subbox was 5×6. Fig. 5(a) displays the real and imaginary parts of the true transfer function for input (2ia) and output (2ob), together with the computed approximation to that transfer function, plotted versus the frequency in gigahertz. Fig. 5(b) are corresponding plots of the $\log_{10} abs(Error)$. We observe that, for this example, the above approximations are very good.

Example 3: Loop over Ground Plane: In this example, we consider a loop over a ground plane as shown in Fig. 6. The maximum time delay is given by the ground plane and equals 0.5963 ns. We applied a 1-mA current source at the terminal as shown and computed the voltage difference there.

Ω was chosen as $[-3, 1] \times [-0.1, 60]$. We compare the magnitude of the voltage response computed using a conventional frequency-domain analysis with the corresponding MOR results (see Fig. 7). We observe close agreement between the two approaches.



(a)



(b)

Fig. 8. (a) Angle bars: poles in $[-3, .1] \times [-.1, 100]$. (b) Two loops: poles in $[-15, 0.1] \times [-0.1, 600]$.

VIII. Stability of Reduced-Order Models

There are two important issues with respect to the stability of reduced-order PEEC models: 1) the stability of the original PEEC model and 2) the fact that MOR procedures may generate reduced-order systems with unstable poles, e.g., [4], [22]. It was shown in [32] that the stability of the original PEEC model can be impacted by model parameters. For very small problems, an analytical approach can be used to gain insight into the stability of delay differential equations for the solution in the time domain [33]. Improvements in time-domain computations have been suggested in [34], where it was shown that the discretization has a strong impact on the stability.

The MOR procedure described in Section VI can be used to study the stability of large PEEC models. The inner-loop iterative method generates poles of the PEEC system. Pole plots can be generated to study the stability of such models with or

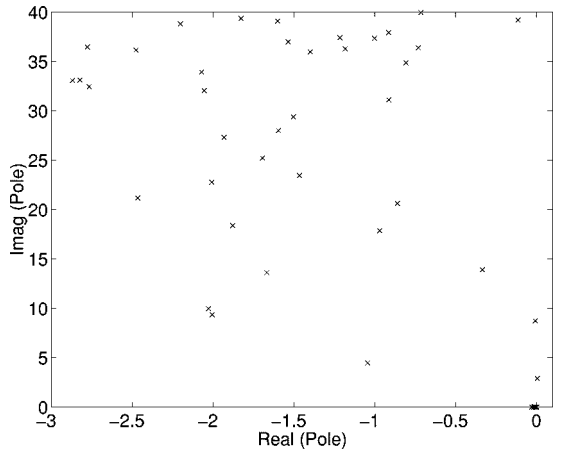


Fig. 9. Loop over ground plane: poles in $[-3, 0.1] \times [-0.1, 40]$.

without retardation for various types of discretizations and modifications within the PEEC models. Fig. 8 display the poles obtained for Examples 1 and 2 in the specified regions of interest in the complex plane. Relevant poles for the third example are given in Fig. 9.

The ultimate goal is the inclusion of such reduced-order models in the simulation of circuit packages. To guarantee the stability of a package model composed of reduced-order models of pieces of the package, we also need to consider the passivity of reduced-order PEEC models. This is an issue which needs to be addressed once the stability issues are resolved. Studies of the stability of several variants of PEEC models are given in [35].

IX. Conclusion

Section VI presents the first systematic and practical method for generating reduced-order models of full wave (L_p, P, R, τ) PEEC models with losses and retardation. At each expansion point, the time-consuming matrix factorizations which are required are the same size as the original system. The new procedure marches systematically through the user-specified portion of the complex plane, computing poles and residues associated with the original PEEC system, which are then combined to obtain a reduced-order model for the original system. The examples demonstrate that very good approximations to transfer functions of PEEC models with retardation can be obtained even when the time delays are very large. The procedure as presented can be applied to the reduction of other similar systems which include time delays. There are many open questions regarding the proposed procedure, such as tradeoffs between the order of the polynomial approximations, subbox size, and the sizes of the time delays. Sparsification of the $A(s)$ could be considered as in [10], but is not straightforward for problems with general conductor geometry. The repetition and patterns within the M_j could also be exploited in the implementation of the MOR procedure. Such issues will be explored in later papers.

In Section VIII, we briefly mentioned the ability to study the stability of large systems with time delays. This aspect of the computations is presently being explored for the stability analysis of large delay-differential systems [35].

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